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FROM MEMBERSHIP TO SEPARATION, A SIMPLE CONSTRUCTION

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In [3], Martin Grötschel, László Lovász and Alexander Schrijver use a construction of Dmitrii Yudin et Arkadiĭ Nemirovskiĭ to polynomially separate a point x from a centered bounded convex K using a membership oracle. In this note, we present a natural and simple construction which solve the same problem but for the simpler case of polyhedra. Namely, given a well defined polyhedron P with a non-empty interior, a point $x \notin P$ and a point $a \in int(P)$, using a polynomial number of calls of the membership oracle, we find a facet of P whose supporting hyperplane separates x from P.

Let P be a well defined polyhedron in the n-dimensional space $\mathbb{Q}^n$ of rational vectors, i.e., there is a bound k of the encoding length of any vertex of P. Additionally suppose that P has a non-empty interior, moreover a point a in the interior of P is given. Notice that the encoding length of supporting hyperplanes of the facets of P and the encoding length of the vertices of P are polynomially equivalent. As in [3], we will denote by $\langle P \rangle$ an upperbound of the encoding length of the elements (vertices or facets) of P, by $int(P)$ the interior of P, and by $bd(P)$ the boundary of P.

In [3], M. Grötschel, L. Lovász and A. Schrijver present a polynomial construction of a separating plane using a membership oracle. Namely, given a bounded, centered convex set K with non void interior, for which the radii of inscribed and circumscribed balls, and a point $x \notin K$ are known, they find a plane separating x and K. This construction use twice the ellipsoid algorithm and a polarity argument. In the simpler case of polyhedra we propose an alternative *natural* construction.

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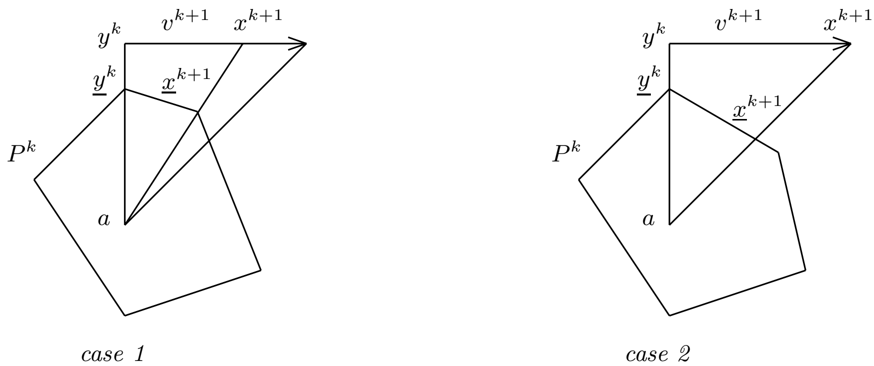

Fig. 1. Construction of x^{k+1}

In the sequel, for a point $y \notin P$ we will denote by $\underline{y}$ the intersection of the segment $[a,y]$ with the boundary of P. So $\underline{y}$ is the central projection with center a of y on the boundary of P.

Let x be a point not in P. We will construct a facet F of P containing the point $\underline{x}$. The supporting hyperplane of F separate x and P.

Denote by G the hyperplane perpendicular to the line (a,x) and passing through the point x. Trivially, the equation of G is polynomially described in the data length. We will assume that our system of coordinates is composed of the direction (a,x) and a system of coordinates of G. Moreover, in order to avoid confusion between affine and linear independences for points of $bd(P)$, we additionally assume that a is the origin of coordinates. The coordinate of the direction (a,x) is named 1, and the unitary vector in the kth direction in G is denoted by v^k. The distance between x and P is of polynomial length, therefore we can choose a unitary length for the coordinate system in G so that the points of G we subsequently construct will stay outside P.

Next we define two pairs of related sequences of points, $\{\underline{x}^i, \underline{y}^i\}$ and $\{x^i, y^i\}$, $i \in \{1, \ldots, n\}$. The points x^i, y^i belong to the plane G, while $\underline{x}^i$, $\underline{y}^i$ are their central projection with center a on the boundary of P. Initially set $y^1 = x^1 = x$ and $\underline{y}^1 = \underline{x}^1 = \underline{x}$. Suppose that for $i \leq k$ we have constructed the points $\underline{x}^i$ and $\underline{y}^i$. Now, let us define the new points $\underline{x}^{k+1}$ and $\underline{y}^{k+1}$.

Definition of $\underline{x}^{k+1}$ and $\underline{y}^{k+1}$:
The two–dimensional plane spanned by the point a, the straight line (a, y^k) and the direction v^{k+1} intersects the polyhedron P along a polygon P^k.

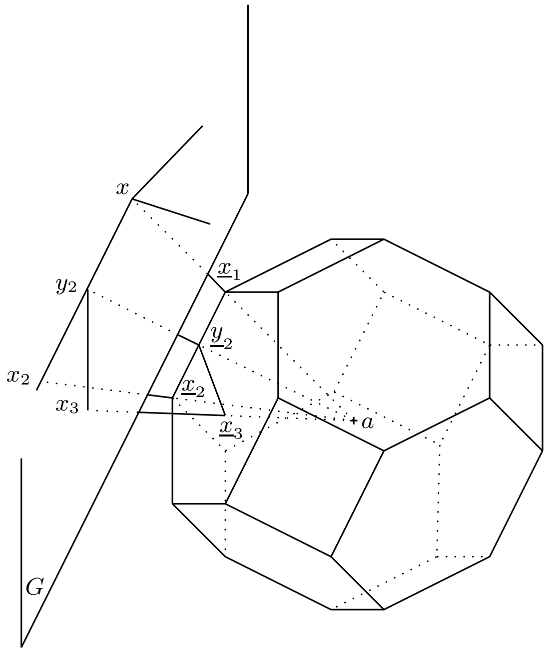

Fig. 2. General construction

Consider the intersection I^k of P^k with the triangle $T^k = (a, y^k, y^k + v^{k+1})$, whose boundary is oriented by the vector v^{k+1}.

The point $\underline{x}^{k+1}$ is the first vertex of I^k, distinct from $\underline{y}^k$, in the direction of v^{k+1} (see Figure 1). We distinguish two cases:

Case 1: the point $\underline{x}^{k+1}$ is a vertex of the polygon P^k not on the boundary of the triangle T^k (this is the case of points x_2 and $\underline{x}_2$ in Figure 2);

Case 2: the point $\underline{x}^{k+1}$ is on the boundary of the triangle T^k (see the points x_3 and $\underline{x}_3$ in Figure 2).

Set $\underline{y}^{k+1} := \frac{1}{2}(\underline{y}^k + \underline{x}^{k+1})$.

Theorem. *The points $\underline{x}^1, \underline{x}^2, \ldots, \underline{x}^n$ span a supporting hyperplane H_F of a facet F containing the point $\underline{x}^1 = \underline{x}$, and thus separating x and the polyhedron P. These points and the hyperplane H_F can be obtained after a polynomial number of calls of the membership oracle.*

Proof. First notice that the section of P with the two-dimensional plane spanned by the points a and y^k, and the direction v^{k+1}, is the polygon P^k, therefore the points $\underline{y}^k$ and $\underline{x}^{k+1}$ are distinct. Notice also that the direction

v^{k+1} in the hyperplane G is independent of the previous ones, thus the points $x^1,x^2,\dots,x^n$ are affinely independent. Since the points $\underline{x}^1,\underline{x}^2,\dots,\underline{x}^n$ belong to a facet F of P, they belong to the supporting hyperplane H_F of this facet. Moreover, $\underline{x}^i=[x^i,a]\cap H_F$, therefore these points are affinely independent as well.

Every face of P which contains a point inside a simplex $S\subseteq P$, also includes S itself. Thus a face which contains both $\underline{y}^k$ and $\underline{x}^{k+1}$ also contains all the $\underline{x}^i$ for $i\le k+1$. Thus the points $\underline{x}^i$ for $i=1,2,\dots,n$ span the hyperplane H which supports the facet F containing $\underline{x}$, establishing the first assertion.

For the second part let me show that:

Lemma. *The points x^i are of polynomial encoding length. Thus, so are the points $\underline{x}^i$.*

Proof. The intersection of the hyperplane G and the polyhedral cone with origin a, basis F, and bounded by below by the hyperplane parallel to G and passing through a, is a well defined polyhedron P_F in G. The central projection on the hyperplane H of the basis of the space defined by the hyperplane G is obviously a basis of H. However, if we change the origin of the basis in G, the corresponding basis in H will change. Therefore we will perform our construction simultaneously in G and in H, the analysis being easier in G. In the polyhedron P_F, let us express x^{k+1} as a function of y^k. We have:

- either $x^{k+1}=y^k+\lambda v^{k+1}$,
- or $x^{k+1}=y^k+v^{k+1}$, and this point has the same encoding length as y^k.

In the first case, the value of λ is chosen so that the point $y^k+\lambda v^{k+1}$ belongs to a facet of P_F. Let $\sum_{j=2}^n \alpha_j z_j = \beta$ be the equation of the supporting hyperplane of this facet written with integer coefficients. Then we have

$$\sum_{j=2}^{n} \alpha_j (y_j^k + \lambda v_j^{k+1}) = \beta.$$

Since v^{k+1} is a unitary vector, we have:

$$\lambda = \frac{1}{\alpha_{k+1}} \left(\beta - \sum_{j=2}^{n} y_j^k \right).$$

Only the $(k+1)^{th}$ component of x^{k+1} is different from the one of y^k:

$$x_{k+1}^{k+1} = \frac{1}{\alpha_{k+1}} \left(\beta - \sum_{j=2}^{n} \alpha_j y_j^k \right).$$

For y^{k+1} the components are again divided by 2, which remains polynomial. The last component will thus be divided by 2^n times the product of the α_i (not of the same hyperplane) which remains polynomial. The $\underline{x}^i$ which are centrally projections of the x^i on the hyperplane H_F are thus also of polynomial length. ∎

It remains to prove that we can effectively construct, in polynomial time, the points $\underline{x}^i$ and $\underline{y}^i$.

The point x^i being given, by dichotomy on the segment $[a,x^i]$ using the membership oracle, we can polynomially approximate $\underline{x}^i$. Now, using a reduction by continuous fractions on each component we can calculate the exact value of $\underline{x}^i$.

Let z be on the hyperplane G outside of P and let z' be the middle of the segment $[y^k,z]$. We can test, in polynomial time, if the segment $[\underline{y}^k,\underline{z}]$ lies on the boundary of P. For this, we have just to check if the points $\underline{z}'$ and the intersection of the lines (a,z') and $(\underline{y}^k,\underline{z})$ are the same. By dichotomy on the segment $[y^k,y^k+v^{k+1}]$, and using the previous construction we can approximate x^{k+1}, and, using continuous fractions again, we can compute the point x^{k+1}. ∎

Remark. We know that for P the hypothesis of being centered and with non void interior are necessary [3]; in the worst case, for a polyhedron in the family $\left[\left(\frac{i}{n},0\right),\left(\frac{n-i}{n},1\right)\right]$ (i being known only by the oracle) we cannot find a point different from $\left(\frac{1}{2},\frac{1}{2}\right)$ with less than n calls to the oracle, which is not polynomial in the encoding length $\log n+\log i$ of this polyhedron.

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